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Lagrangians and Hamiltonians for One-Dimensional Autonomous Systems

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An equation is obtained to find the Lagrangian for a one-dimensional autonomous system. The continuity of the first derivative of its constant of motion is assumed. This equation is solved for a generic nonconservative autonomous system that has certain quasi-relativistic properties. A new method based on a Taylor series expansion is used to obtain the associated Hamiltonian for this system. These results have the usual expression for a conservative system when the dissipation parameter goes to zero. An example of this approach is given.

KEY WORDS: Lagrangian; Hamiltonian; constant of motion; nonconservative autonomous system.

1. INTRODUCTION

Lagrangians and Hamiltonians occupy an important position in the development of physics. Modern physics theories are formulated in terms of Hamiltonian or Lagrangian structures. Very often the Lagrangian of a system can be used to find its constants of motion which can give insight into the stability and periodicity of the system (Vujanovic and Jones, 1989). For an autonomous system the Hamiltonian itself is a constant of motion of the system.

When a system is conservative, the Lagrangian and the Hamiltonian can be obtained by subtracting or adding respectively the kinetic and potential energy of the system (Goldstein, 1980). The ease with which such constructions are made has contributed to their immense popularity in the field of physics. However, this construction is not useful for finding Lagrangians and Hamiltonians for nonconservative systems. The reason is that there is not yet a consistent Lagrangian and Hamiltonian formulation for nonconservative systems. The problem of obtaining the Lagrangian and Hamiltonian from the equations of motion of a mechanical system is a particular case of "The Inverse problem of the Calculus of Variations" (Santilli, 1978). This topic has been studied by many mathematicians and theoretical physicists since the end of the last century. The interest of physicists

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in this problem has grown recently because of the quantization of nonconservative systems. A mechanical system can be quantized once its Hamiltonian is known and this Hamiltonian is usually obtained from a Lagrangian (Um *et al.*, 2002).

The problem of the existence of a Lagrangian for one-dimensional systems was solved by Darboux (Darboux, 1984), and the relationship between the constant of motion and the Lagrangian for one-dimensional autonomous systems was given by Kobusen-Leubner-López (Kobussen, 1979; Leubner, 1981; López, 1996). The problem arises when one tries to obtain the Hamiltonian expressing the velocity in terms of the canonical variables, which is not possible to do in general. The main purpose of this paper is to obtain the Lagrangian and the Hamiltonian for a nonconservative autonomous system given by $mdv/dt = (-dU/dx + \gamma(x)v^2)(1 - \alpha^2 v^2)$ where $U(x)$ is the potential energy, v is the velocity, $\gamma(x)$ is an arbitrary function of position and α^2 is any real number. This mechanical system is of interest because it represents, at first order of approximation, the motion of a relativistic particle under the action of a dissipative force which is proportional to the square of the velocity.

2. CONSTANT OF MOTION, LAGRANGIAN AND HAMILTONIAN

Newton's equation of motion for one-dimensional autonomous systems can be written as the following dynamical system

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = F(x, v), \tag{1}$$

where x is the position of the particle, v is the velocity and $F(x, v)$ is the force divided by the mass of the particle. Let $K = K(x, v)$ be a constant of motion of (1), then

$$v\frac{\partial K}{\partial x} + F(x, v)\frac{\partial K}{\partial v} = 0. \tag{2}$$

Assuming the following condition over the constant of motion

$$\frac{\partial^2 K}{\partial x \partial v} = \frac{\partial^2 K}{\partial v \partial x}, \tag{3}$$

and using the fact that for one-dimensional autonomous systems a constant of motion is given in terms of the Lagrangian by (Goldstein, 1980)

$$K(x, v) = v\frac{\partial L}{\partial v} - L, \tag{4}$$

then (3) leads to

$$v\frac{\partial G}{\partial x} + \frac{\partial (F\,G)}{\partial v} = 0, \tag{5}$$

where the Euler–Lagrange equation (Goldstein, 1980) has been used and $G = \partial^2 L/\partial v^2$. Once a nontrivial solution for G has been found, the Lagrangian is obtained through the integration

$$L(x, v) = \int dv \int G(x, v)\, dv + f_1(x)v - f_2(x), \tag{6}$$

where $f_1(x)$ and $f_2(x)$ are arbitrary functions. The second term on the right side of (6) corresponds to a gauge of the Lagrangian which brings about an equivalent Lagrangian (Goldstein, 1980), and it is possible to forget it. The function $f_2(x)$ can be determined as it will be showed for the following mechanical system given by

$$m\frac{dv}{dt} = \left(-\frac{dU}{dx} + \gamma(x)v^2\right)(1 - \alpha^2 v^2), \tag{7}$$

where $U(x)$ is the potential energy, v is the velocity, $\gamma(x)$ is an arbitrary function of position and α^2 is any real number. It is easy to convince oneself that a solution to (5) for the mechanical system (7) is

$$G(x, v) = \frac{m}{(1 - \alpha^2 v^2)^2} \exp\left[-\frac{2}{m}\left(\int \gamma(x)\, dx - \alpha^2 U(x)\right)\right], \tag{8}$$

using (8) one gets the generalized linear momentum

$$p = \frac{m}{2}\left(\frac{v}{1 - \alpha^2 v^2} + \frac{\tanh^{-1}(\alpha v)}{\alpha}\right) \exp\left[-\frac{2}{m}\left(\int \gamma(x)\, dx - \alpha^2 U(x)\right)\right], \tag{9}$$

and the Lagrangian

$$L(x, v) = \frac{mv}{2\alpha} \tanh^{-1}(\alpha v) \exp\left[-\frac{2}{m}\left(\int \gamma(x)\, dx - \alpha^2 U(x)\right)\right] - f_2(x). \tag{10}$$

To determine the function f_2, one uses the Euler–Lagrange equation (Goldstein, 1980)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x}. \tag{11}$$

Substituting (10) into (11) one gets the following equation for $f_2(x)$

$$\frac{df_2}{dx} = \frac{dU}{dx} \exp\left[-\frac{2}{m}\left(\int \gamma(x)\, dx - \alpha^2 U(x)\right)\right], \tag{12}$$

integrating equation (12) with respect to x the function $f_2(x)$ is obtained and the constant of motion is given by

$$K(x, v) = \frac{mv^2}{2(1 - \alpha^2 v^2)} \exp\left[-\frac{2}{m}\left(\int \gamma(x)\, dx - \alpha^2 U(x)\right)\right] + f_2(x). \tag{13}$$

To obtain the Hamiltonian of the system, we express the constant of motion in terms of the position and the generalized momentum

$$H(x, p) \equiv K(x, v(x, p)), \tag{14}$$

therefore one has to solve (9) for the velocity as a function of the position and the generalized momentum, which at first sight may seem like a formidable task but it can be done if we restrict ourselves to the case $|\alpha v| < 1$, as it will be showed in the following example.

3. EXAMPLE

Consider a relativistic particle of mass m at rest under the action of a constant force $\lambda > 0$ and immersed in a medium that exerts some type of friction which is proportional to the square of the velocity. The classical equation of motion for this system is given by

$$m\frac{dv}{dt} = (\lambda - \gamma v^2)(1 - v^2/c^2)^{3/2}, \tag{15}$$

where γ is a positive real parameter and c represents the peed of light. Writing (15) at first order of approximation in v^2/c^2 we have

$$m\frac{dv}{dt} = (\lambda - \gamma v^2)(1 - \alpha^2 v^2), \tag{16}$$

where $\alpha^2 = \frac{3}{2c^2}$. The mechanical system (16) is a particular case of the mechanical system given by (7) where $U(x) = -\lambda x$ and $\gamma(x) = -\gamma$. Using (8)–(10) we have

$$G(x, v) = \frac{m}{(1 - \alpha^2 v^2)^2} \exp\left[-\frac{2x}{m}\left(\lambda\alpha^2 - \gamma\right)\right], \tag{17}$$

$$p = \frac{m}{2}\left(\frac{v}{1 - \alpha^2 v^2} + \frac{\tanh^{-1}(\alpha v)}{\alpha}\right) \exp\left[-\frac{2x}{m}\left(\lambda\alpha^2 - \gamma\right)\right], \tag{18}$$

$$L(x, v) = \frac{mv \tanh^{-1}(\alpha v)}{2\alpha} \exp\left[-\frac{2x}{m}\left(\lambda\alpha^2 - \gamma\right)\right] - f_2(x), \tag{19}$$

to find $f_2(x)$ we have to solve the equation

$$\frac{df_2}{dx} = -\lambda \exp\left[-\frac{2x}{m}\left(\lambda\alpha^2 - \gamma\right)\right], \tag{20}$$

which has the following solution

$$f_2(x) = \frac{m\lambda}{2(\lambda\alpha^2 - \gamma)}\left(e^{-2x(\lambda\alpha^2 - \gamma)/m} - 1\right). \tag{21}$$

Therefore, one obtains the Lagrangian

$$L(x,v) = \frac{mv\tanh^{-1}(\alpha v)}{2\alpha}e^{-2x(\lambda\alpha^2-\gamma)/m} - \frac{m\lambda}{2(\lambda\alpha^2-\gamma)}\left(e^{-2x(\lambda\alpha^2-\gamma)/m}-1\right), \tag{22}$$

and the constant of motion

$$K(x,v) = \frac{mv^2}{2(1-\alpha^2v^2)}e^{-2x(\lambda\alpha^2-\gamma)/m} + \frac{m\lambda}{2(\lambda\alpha^2-\gamma)}\left(e^{-2x(\lambda\alpha^2-\gamma)/m}-1\right). \tag{23}$$

To obtain the Hamiltonian of the system, we restrict ourselves to the case $|\alpha v| < 1$, therefore equation (18) reads

$$pe^{2x(\lambda\alpha^2-\gamma)/m} = \frac{m}{2}\sum_{n=0}^{\infty}\left(\frac{2n+2}{2n+1}\right)\alpha^{2n}v^{2n+1}, \tag{24}$$

comparing both sides of (24) we conclude that

$$v^{2n+1} = \frac{2n+1}{(n+1)!}\left(\frac{2\lambda x}{m}\right)^n \frac{p}{m}e^{-2\gamma x/m}, \tag{25}$$

therefore, the Hamiltonian is given by

$$H(x,p) = \frac{m}{2}e^{-2x(\lambda\alpha^2-\gamma)/m}\sum_{n=0}^{\infty}\alpha^{2n}v^{2n+2}(x,p) + \frac{m\lambda}{2(\lambda\alpha^2-\gamma)}\left(e^{-2x(\lambda\alpha^2-\gamma)/m}-1\right), \tag{26}$$

where $v^{2n+2}(x,p)$ is given by

$$v^{2n+2}(x,p) = \left(\frac{p}{m}e^{-2\gamma x/m}\frac{2n+1}{(n+1)!}\left(\frac{2\lambda x}{m}\right)^n\right)^{(2n+2)/(2n+1)}. \tag{27}$$

Notice that the Hamiltonian (26) has physical meaning only when $p > 0$ and $x > 0$. Using Hamilton's equations of motion we have

$$\dot{x} = e^{-2\lambda\alpha^2x/m}\sum_{n=0}^{\infty}\frac{(2\lambda\alpha^2x/m)^n}{n!}\left[\frac{(2n+1)pe^{-2\gamma x/m}}{(n+1)!m}\left(\frac{2\lambda x}{m}\right)^n\right]^{1/(2n+1)}, \tag{28}$$

$$\dot{p} = e^{-2x(\lambda\alpha^2-\gamma)/m}\left[\lambda + (\lambda\alpha^2-\gamma)\sum_{n=0}^{\infty}\alpha^{2n}\left[\frac{(2n+1)pe^{-2\gamma x/m}}{(n+1)!m} \times\left(\frac{2\lambda x}{m}\right)^n\right]^{(2n+2)/(2n+1)}\right]$$

$$- \frac{2p}{m} e^{-2\lambda\alpha^2 x/m} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \left[-\gamma \left(\frac{2\lambda x}{m} \right)^n + \lambda n \left(\frac{2\lambda x}{m} \right)^{n-1} \right]$$

$$\times \left[\frac{(2n+1) p e^{-2\gamma x/m}}{(n+1)! m} \left(\frac{2\lambda x}{m} \right)^n \right]^{1/(2n+1)} .$$

Hamilton's first equation of motion (28) give us the velocity as a function of the position and the generalized momentum for the case $|\alpha v| < 1$. All the expressions derived have the right limit when $\gamma \to 0$ and $\alpha \to 0$.

4. CONCLUSIONS

The general form of the Lagrangian, the constant of motion and the generalized momentum were obtained for the following nonconservative autonomous system $m\, dv/dt = (-dU/dx + \gamma(x)v^2)(1 - \alpha^2 v^2)$. The Hamiltonian associated to the system was found for the case $|\alpha v| < 1$. All the expressions obtained in this paper converge to the conservative case when the dissipation parameter goes to zero.

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